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THE DECAY OF AN ARBITRARY INITIAL DISCONTINUITY IN AN ELASTIC MEDIUM*

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The solution of the problem of the decay of an arbitrary discontinuity in elastic theory is studied. It is assumed that a plane boundary separates an elastic homogeneous, non-heat-conducting medium into two half-spaces with different elastic properties and densities. Each of the media possesses an arbitrary kind of homogneous initial strain (stress) and velocity. In the sequel the stresses and velocities of the media are assumed to be continuous at the boundary. This results in the formation of a system of plane selfsimilar waves (simple and shock), which propagate in each of the half-spaces. The problem is solved under the assumption of weak non-linearity and anisotropy of the materials. This permits an approximate evaluation of the stress and strain at the contact discontinuity. After this the problem on the decay of an arbitrary initial discontinuity is reduced to two problems on the sudden change of load on a half-space boundary, which are solved independently for each of the media.

1. Fundamental equations.

We assume the plane $x = 0$ in the Lagrange variables $x_1, x_2, x_3 = x$ to separate two media. In the unstressed state, the axes x_i form a rectangular Cartesian system in each of the media, where the axes x_1, x_2 in the plane of the boundary can be oriented differently in the media to the right and left of the interfacial boundary. We denote the medium on the right of the boundary $(x > 0)$ by A and the medium on the left $(x < 0)$ by B. Each of the media possess its homogeneous initial strain $\varepsilon_{ij}^{A,B} = \text{const}$ and initial velocity $V_i^{A,B} = \text{const}$. The problem is selfsimilar.

We assign its internal energy U or elastic potential $\Phi = \rho_0 U$ to each of the elastic media. Let $\Phi^A = \rho_0^A U^A (\partial w_i/\partial x_j, g, S)$. for the medium A $(x > 0)$ while $\Phi^B = \rho_0^B U^B (\partial w_i/\partial x_j, g, S)$ for the medium B $(x < 0)$. Here ρ_0 is the density of the medium in the unstressed state, w_i are the components of the displacement vector, S is the entropy per unit mass, and g is a parameter characterizing the anisotropy of the media, more truly, giving definite symmetry properties that the material possesses. The parameters g can generally be scalars, vectors, or tensors.

We consider them to be constants that do not change during the passage of the strain wave.

Only the $\partial w_i/\partial x$ of the strain gradient components $\partial w_i/\partial x_j$ vary in plane waves, for which it is convenient to introduce the notation $\partial w_i/\partial x = u_i (x, t)$ $(i = 1, 2, 3)$. All the rest $\partial w_i/\partial x_\alpha$ $(\alpha = 1, 2)$ remain constants, where we set $\partial w_3/\partial x_\alpha = 0$, so that the wave fronts are shifted parallel to the initial location of the interfacial boundary. The rest $\partial w_\alpha/\partial x_\beta = \text{const}$ can be produced by deformation anisotropy of the medium.

One-dimensional adiabatic motions (plane waves) in each of the media A and B on different sides of the boundary $x = 0$ are described by the non-linear system of equations /1, 2/

$$\rho_0^{A,B} \frac{\partial v_i}{\partial t} = \frac{\partial \sigma_{i3}}{\partial x}, \quad \sigma_{i3} = \frac{\partial \Phi^{A,B}}{\partial u_i} \tag{1.1}$$

$$\frac{\partial v_i}{\partial x} = \frac{\partial u_i}{\partial t}, \quad \frac{\partial S}{\partial t} = 0, \quad i = 1, 2, 3$$

Here $v_i = \partial w_i/\partial t$ are velocity vector components, and σ_{i3} are the Piola-Kirchhoff stress tensor components.

We consider that slip does not occur on the interfacial boundary of the two media, it is a contact discontinuity on which the velocity and stress vectors are continuous. It is convenient to write the boundary conditions separately for the normal and tangential components of the designated vectors. The subscript τ means that the vector in question lies in a plane parallel to the wave front

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$$\mathbf{v}_\tau{}^A = \mathbf{v}_\tau{}^B, \quad \sigma_{\tau 3}^A = \sigma_{\tau 3}^B \tag{1.2}$$

$$v_3{}^A = v_3{}^B, \quad \sigma_{33}{}^A = \sigma_{33}{}^B \tag{1.3}$$

The initial states of the media A and B (for $t = 0$) are given by the strain components $u_i\ (t = 0) = U_i^{A,B}$, $(\partial w_\alpha/\partial x_\beta)^{A,B}$ $(i = 1, 2, 3;\ \alpha, \beta = 1, 2)$ of the velocity $v_i\ (t = 0) = V_i^{A,B}$ and the entropy $S^{A,B}$. Moreover, the media are characterized by constant anisotropy parameters $g^{A,B}$.

2. An approximate approach. We will consider both elastic media to possess weak non-linearity so that the solution can be obtained by expansion in a small parameter ε (the characteristic magnitude of the strain). Both media can have weak anisotropy, natural or due to strain. We take its characteristic magnitude as δ. The presence of a small non-linearity and anisotropy transforms the longitudinal and transverse waves into quasilongitudinal and quasi-transverse. Both the non-linearity and the anisotropy will be taken into account in just the principal terms disclosing the deviation of the solution from a linear isotropic one. An investigation of simple and shock waves in such an approximate formulation /2-6/ showed that the quasilongitudinal and quasitransverse waves can be studied independently, where it is sufficient to represent the elastic potential of the medium by an expansion in the strain components to terms of order ε^3 for the quasilongitudinal waves and to terms of order ε^4 for the quasitransverse waves in order to clarify the principal non-linear effects.

Only quasitransverse waves will be considered as the most interesting and complex resulting in new effects (as compared with gas dynamics). Non-linear effects not taken into account in the expansion of Φ for them, differ from those written down by not less than ε^2. We also take the anisotropy into account by its principal terms, the lowest in the expansion in δ, namely, linear in δ. As is shown in /6/, the expansion of the potential Φ for quasitransverse waves in a specially selected coordinate system u_1, u_2 is in even powers of u_α, i.e., has the form /6, 7/

$$\begin{gathered}\Phi = {}^1/_2\,(f - g)\,u_1{}^2 + {}^1/_2\,(f + g)\,u_2{}^2 - \varkappa\,(u_1{}^2 + u_2{}^2)^2 + \\ \rho_0 T_0\,(S - S_0) \\ f = \mu + O\,(\varepsilon) = \rho_0 c^2\end{gathered} \tag{2.1}$$

for a broad class of anisotropic media (including transversally isotropic and orthotropic). Here c is the characteristic velocity of linear isotropic transverse waves, μ is the Lame coefficient of an elastic medium, $\varkappa$ is still another elastic modulus of the medium that introduces non-linear effects, and the constant $g \sim \delta$ describes the anisotropy. The relative error of the expansion taken does not exceed the small quantity $\chi = \max\{\varepsilon^2, \delta\}$, i.e., terms not taken into account differ from those written by not less than the quantity χ.

We note that because of the fact that the form (2.1) for writing the potential Φ in each of the media is obtained in specially selected coordinate axes u_1, u_2, these axes can be oriented differently in the media A and B and can have a different origin.

To eliminate a quasilongitudinal wave from consideration, we consider the initial values of the normal stress and velocity components to the boundary identical to accuracy χ in both media. This enables us (to the same relative accuracy χ) to consider the group of boundary conditions (1.3) as satisfied.

3. Determination of the state on the contact discontinuity. We reduce the problem of decay of the initial discontinuity to a selfsimilar problem on the sudden change of the stresses on the half-space boundary. To do this, values of the strain $u_\alpha\ (\alpha = 1, 2)$ must be determined on the contact boundary for $t > 0$. Because of selfsimilarity these will be constant quantities u_α^* which will differ for the media A and B in the general case.

As is shown in /7/, for waves propagating to one side, the velocity and strain components in a wave can be connected by the same relationships as in a linear isotropic wave (the relative error here does not exceed the quantity χ)

$$v_\alpha - V_\alpha{}^A = -c^A\,(u_\alpha - U_\alpha{}^A) + O\,(\varepsilon\chi) \ \text{ for waves going to the right} \tag{3.1}$$

$$v_\alpha - V_\beta{}^B = c^B\,(u_\alpha - U_\alpha{}^B) + O\,(\varepsilon\chi) \ \text{ for waves going to the left}$$

Using (2.1) and (3.1), we write down only the principal terms explicitly for quasi-transverse waves under conditions on the contact surface of the media $(x = 0)$

$$\begin{gathered}-c^A\,(\mathbf{u}_\tau^{*A} - \mathbf{U}_\tau{}^A) + \mathbf{V}_\tau{}^A = c^B\,(\mathbf{u}_\tau^{*B} - \mathbf{U}_\tau{}^B) + \mathbf{V}_\tau{}^B + O\,(\varepsilon\chi) \\ \mu^A\mathbf{u}_\tau^{*A} = \mu^B\mathbf{u}_\tau^{*B} + O\,(\varepsilon\chi)\end{gathered} \tag{3.2}$$

The values $\mathbf{u}_\tau^{*A,B}$ found from this system on the contact plane will be determined to the accuracy of χ. But, as mentioned above, the simple and shock waves /2-5/ were investigated to precisely such accuracy and we shall use it to construct the solution of the problem of the decay of a discontinuity. There is a solution of the selfsimilar problem on a sudden change in the load on a half-space boundary /8, 9/ with the same relative error χ, where domains of values of u_α^* are indicated to the same accuracy χ on the boundary, which results in different forms of the solution. Consequently, a linear approximation in writing the conditions on the boundary and their resolution is sufficient for a qualitative investigation of the problem of the decay of a discontinuity.

Thus, solving (3.2) for u_τ^*, we obtain

$$\mu^A\mathbf{u}_\tau^{*A} = \mu^B\mathbf{u}_\tau^{*B} = \frac{\mu^A\mu^B}{\mu^Ac^B+\mu^Bc^A}(c^A\mathbf{U}_\tau^A + c^B\mathbf{U}_\tau^B + \Delta\mathbf{V}) \tag{3.3}$$
$$c^{A,B} = (\mu^{A,B}/\rho_0^{A,B})^{1/2}, \quad \Delta\mathbf{V} = \mathbf{V}_\tau^A - \mathbf{V}_\tau^B$$

When the medium on both sides of the plane $x=0$ is identical, i.e., ρ_0 and μ are identical, but the difference is just in the initial strains $U_\tau^{A,B}$ and the velocities $V_\tau^{A,B}$ the vectors u_τ^* on the boundary are continuous and in place of (3.3) we have a simple expression for them

$$\mathbf{u}_\tau^{*A} = \mathbf{u}_\tau^{*B} = \mathbf{U}^\circ + {}^1\!/_2\Delta\mathbf{V}/c, \quad \mathbf{U}^\circ = {}^1\!/_2(\mathbf{U}_\tau^A + \mathbf{U}_\tau^B) \tag{3.4}$$

4. The problem for a half-space. Now, for each of the half-spaces to the right and left of $x=0$ we have the following selfsimilar problem: for $t=0, x>0$ and $x<0$ each medium possesses its own strain $U_\alpha^{A,B}$, at the time $t=0$ the strains on the boundary $x=0$ take the value $u_\alpha^{*A,B}$ and remain such later. The problems are solved independently for each of the half-spaces. The solution of such a problem, to the accuracy assumed here, is worked out in /8, 9/. Systems of selfsimilar simple and shock waves proceed on both sides of the boundary. Their configurations in terms of u_α^* depend on the initial velocity $\mathbf{V}_\tau$ and strain $\mathbf{U}_\tau$ of the media. For fixed $\mathbf{U}_\tau^A$, $\mathbf{U}_\tau^B$ the boundary values $\mathbf{u}_\tau^*$ depend linearly on the vector of the relative velocity $\Delta\mathbf{V}$. For each point of the plane $\Delta V_1, \Delta V_2$ a $\mathbf{u}_\tau^{*A,B}$ can be found, and therefore, by using /8, 9/ those wave systems can be indicated that propagate on both sides of the boundary.

5. The problem of the decay of a discontinuity in an isotropic medium. As an example, we consider the simplest case of a problem when a medium is identical on both sides and without anisotropy ($g^A = g^B = 0$). Then part of the shock adiabat of the quasi-transverse shockwaves (circle) coincides with one of the integral curves of the simple waves. These waves propagate with a change in shape and entropy. By analogy with magnetohydrodynamics, we call them a rotational discontinuity. In media where the coefficient $\varkappa$ in (2.1) is positive, the rotational discontinuity is a fast wave /2-5/ and it proceeds in front of other waves on both sides of the boundary $x=0$. In media with $\varkappa<0$ the rotational discontinuity is a slow wave, the trailing wave in the sequence. To be specific let $\varkappa>0$. Which wave follows behind the rotational discontinuity depends on what domain of the plane u_α the point u_α^* imaging the state on the contact discontinuity would be incident on.

The initial states U_α^A and U_α^B are marked by the points A and B in Fig.1. For $\Delta\mathbf{V}=0$ the boundary values u_α^* yield the point $M^\circ(U_\alpha^\circ)$ according to (3.4). Other magnitudes of the relative velocity difference $\Delta V \neq 0$ can yield another point of the plane u_α as the state u_α^* on the contact boundary. Two concentric circles passing though the initial points $A\,(U_\alpha^A)$ and $B\,(U_\alpha^B)$ (Fig.1a) divide the plane u_α into three domains. The incidence of u_α^* into any domain will result in different kinds of solution of the problem of discontinuity decay.

For fixed $\mathbf{U}_\tau^A, \mathbf{U}_\tau^B$ the values $\mathbf{u}_\tau^*$ depend on $\Delta\mathbf{V}$. Let us display the ends of the vectors $\Delta\mathbf{V}$ by points of the plane $\Delta V_\alpha/c$ (Fig.1b) and let us divide the whole plane $\Delta V_\alpha/c$ into a domain such that their points yield the value u_α^* in the domains indicated in Fig.1a. Because of the linearity of the dependence of $\mathbf{u}_\tau^*$ on $\Delta\mathbf{V}$ the appropriate domains in the plane $\Delta\mathbf{V}_\alpha/c$ are also formed by concentric circles with centre at the point $O_1(-U_\alpha^\circ)$ and passing through the points $A\,(U_\alpha^A - U_\alpha^B)$ and $B\,(U_\alpha^B - U_\alpha^A)$. When the point u_α^* lies within an annular domain a rotational discontinuity first propagates to the right in the state A and a slow shock behind it. The rotational discontinuity is to the left (in the state B) and a slow simple wave behind it (Fig.2a). Behind the rotational discontinuity for the state u_α^* slow simple waves proceed outside the large circle on both sides, and slow shocks for the state u_α^* within the smaller circle.

The wave systems that can be obtained during decay of the initial discontinuity in an incompressible isotropic elastic medium under definite conditions on the kind of elastic potential are also indicated in /10/.

6. Influence of anisotropy of the medium on the solution. When the medium possesses the anisotropy $(g^A = g^B = g \neq 0)$ on both sides of the interfacial boundary, the geometry of the domains consisting of the boundary values u_α^* resulting in different kinds of solutions becomes more complex. These domains and their corresponding solutions of the problem on waves proceeding from the boundary are displayed in /8/ for media with $\varkappa > 0$ and in /9/ for $\varkappa < 0$. For instance, for the state u_α^* shown in Fig.1 by the point M, when there is no anisotropy we have the solution shown in Fig.2b, and for $g \neq 0$ the solution is displayed in Fig.3.

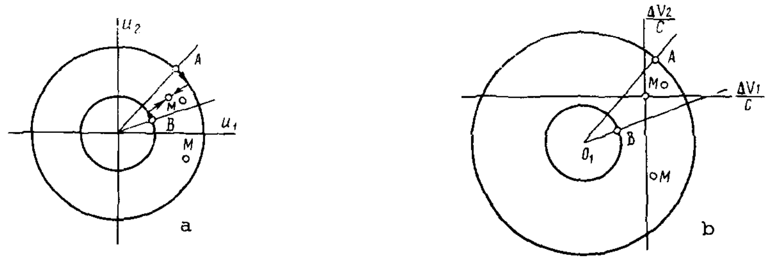

Fig.1

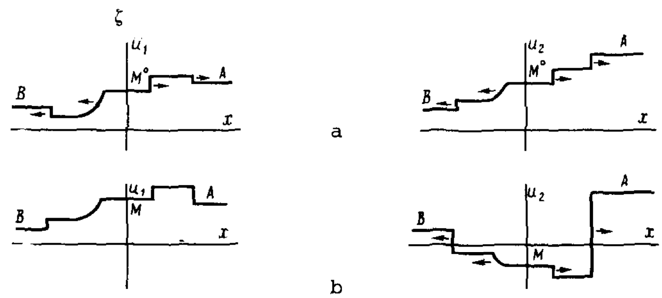

Fig.2

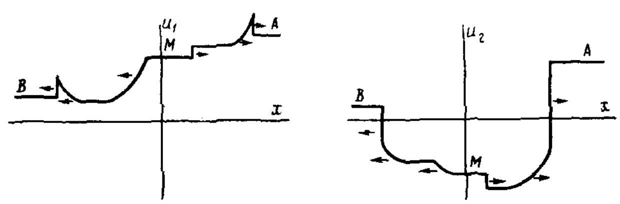

Fig.3

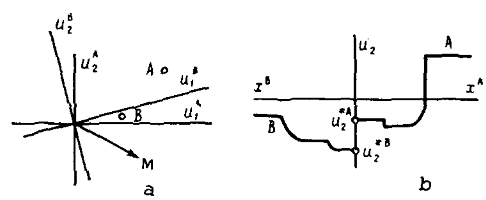

Fig.4

If $g^A \neq g^B$, then for continuity on the boundary of the vector $\mathbf{u}_\tau^{*A} = \mathbf{u}_\tau^{*B}$ because of the different orientation of the axes u_α^A and u_α^B (Fig.4a) the magnitude of the components of this vector for the domain A and the domain B are different, which also induces a contribution to the difference in the kind of solution (Fig.4b).

7. The problem of the collision of shocks. On the basis of the solution obtained the problem of the reflection and refraction of waves at the interfacial boundary of two media, on the collision of shocks, etc., can be investigated. For instance, two shocks proceed, one towards the other, in a medium at rest that possesses the initial deformation U_α°, $e_{\alpha\alpha}$. Let the intensities of these waves be such that the deformed state acquires a component U_α^A behind the wave front going from right to left (in the negative direction of the x axis) while the strain components proceeding oppositely behind the front have the magnitude U_α^B. When these two waves collide new magnitudes for the strain components $u_\alpha^* = U_\alpha^A + U_\alpha^B - U_\alpha^\circ$ should be obtained on the collision boundary.

Since U_α^A, U_α^B are arbitrary, the state u_α^* on the boundary can be imagined as any point in the plane u_α. Furthermore, as before, two problems should be solved for the half-spaces in which the initial deformed state is given by the quantities $U_\alpha^{A,B}$, while on the boundary it suddenly changes to u_α^* and is later retained. The form of the solution is obtained, for instance, just as in one of the Figs.2 and 3.

It is necessary to recall here that according to /1, 3, 5/ shocks when there is no initial deformation U_α° can exist only in media with $\varkappa < 0$, for $U_\alpha^\circ \neq 0$ in others.

Note that the solution of all the problems elucidated is based on the solution of the selfsimilar problem regarding a sudden change in the deformation on the boundary of an elastic half-space. It was found in the solution of this problem /8, 9/ that for certain relationships between the anisotropy and the initial deformations, a domain of values u_α^*, although small, can appear for which the solution is not unique. For these values of u_α^* additional investigations are necessary. Such an investigation is performed in /11/ and enables one to say to which of the two possible solutions preference should be given.

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CONSTRUCTION OF DISCONTINUOUS SOLUTIONS OF THE EQUATIONS OF PLANE ELASTICITY THEORY BY THE METHOD OF GENERALIZED FUNCTIONS*

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A method of constructing integral representations of discontinuous solutions of the equations of plane elasticity theory based on the use of the apparatus of the theory of generalized functions is described. The representations obtained for the discontinuous displacement and stress field components are utilized to formulate sufficient conditions ensuring continuous continuation of these quantities at almost all the points of the line of discontinuity.

1. Formulation of the problem. We consider the complete system of equations of plane elasticity theory describing the state of plane strain of a cylindrical body when there are no mass forces and initial stresses /1/ in a system of rectangular Cartesian coordinates

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